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RELAXATION AT CRITICAL POINTS: DETERMINISTIC AND STOCHASTIC THEORY

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A generalized critical point can be characterized by non-linear dynamics. We formulate the deterministic and stochastic theory of relaxation at such a point. Canonical problems are used to motivate the general solutions. In the deterministic theory, we show that at the critical point certain modes have polynomial (rather than exponential) growth or decay. The stochastic relaxation rates can be calculated in terms of various incomplete special functions. Three examples are considered. First, a substrate inhibited reaction (marginal type dynamical system) is treated. Second, the relaxation of a mean field ferromagnet is considered. We obtain a result that generalizes the work of Griffiths et al. Third, we study the relaxation of a critical harmonic oscillator.

1. Introduction: "Critical slowing down"

Thermodynamic and kinetic generalized critical points are characterized by non-linear dynamics. Such non-linear dynamics lead to many interesting phenomena, e.g., "anomalous" fluctuations (treated in ref. 1) and the "slowing" down of the decay of a perturbation. To illustrate the latter effect, consider the kinetic equation:

$$\dot{x} = b(x, \alpha), \quad x \in R^1, \alpha \in R^n, \tag{1.1}$$

for which the origin is assumed to be a steady state: $b(0, \alpha) = 0$. Suppose that the system is perturbed to a value $x = x_0$. A well defined problem is to calculate the time that the system takes to reach $\delta (0 < \delta \ll x_0)$ from x_0. If x_0 is "small," then a natural approach involves approximating (1.1) by

$$\dot{x} = b'(0, \alpha)x + \mathcal{O}(x^2), \quad x(0) = x_0. \tag{1.2}$$

We assume that $b'(0, \alpha) < 0$. Then the perturbation will decay. The time that the system takes to reach $x = \delta$ is easily calculated to be

$$t_\delta = \frac{1}{b'(0, \alpha)} \ln\left[\frac{\delta}{x_0}\right]. \tag{1.3}$$

However, suppose that for some critical value of $\alpha = \alpha_c$, $b'(0, \alpha_c) = 0$. Then $(0, \alpha_c)$ is a "generalized critical" point: the dynamics at $\alpha = \alpha_c$ are totally non-linear. Eq. (1.3) yields the physically ridiculous result $t_\delta = \infty$. Furthermore, (1.2) becomes $\dot{x} = 0$. Both of these difficulties are due to improper linearization procedures, and not any physical divergences. In fact, the decay of the perturbation is algebraic in time, with the exact form determined by the nature of the singularity at $(0, \alpha_c)$. Such simple problems and the canonical bifurcations are considered in section 2. The points essential to the understanding of critical relaxation phenomena can be gained by study of one-dimensional systems.

If fluctuations are not included, a steady state cannot be attained in finite time. Since the deterministic forces vanish as a steady state or equilibrium is approached, the ratio of fluctuation intensity to deterministic dynamics grows. Thus, the proper theory of relaxation must be a stochastic one. The deterministic kinetic equation can be modified by a Langevin approach. We use the diffusion approximation to treat the stochastic kinetic equation. In particular, we give a diffusion equation for $T(x' \mid x)$, the expected time to reach x', starting at x, conditioned on the fact that the process reaches x'. We analyze the one-dimensional equations fully and obtain certain special functions, which are generalized in section 4 to the solution of multi-dimensional problems. In sections 5–7, we consider three applications of the theory. In section 5, relaxation from a steady state of marginal stability in a substrate inhibited reaction is considered. In section 6, we consider relaxation of a mean field ferromagnet. Our results complement and extend the results of Griffiths et al.[2]). Finally, in section 7, we discuss relaxation phenomena in the critical harmonic oscillator[1,3]).

2. Deterministic theory of relaxation at critical points

In this section, we give the deterministic theory of relaxation at critical points. Our classification scheme extends the ideas of Kubo et al.[4]) to multi-dimensional systems (section 2.2). In section 2.1, we stress the one-dimensional results, because the multi-dimensional theory is a natural extension of the one-dimensional results.

2.1. *One-dimensional systems*

We consider a kinetic equation

$$\dot{x} = b(x, \alpha), \quad x(0) = x_0, \quad x \in R^1, \quad \alpha \in R^n. \tag{2.1}$$

The origin is assumed to be a steady state of (2.1).

2.1.1. *Normal type*

The steady state is of the normal type if $b'(0, \alpha) \neq 0$. It is stable if $b'(0, \alpha) < 0$ and unstable if $b'(0, \alpha) > 0$. In the vicinity of the origin, (2.1) can be replaced by

$$\dot{x} = b'(0, \alpha)x, \quad x(0) = x_0. \tag{2.2}$$

As mentioned in the introduction, the time that the system takes to reach $x = \delta$, starting at $x = x_0$ is

$$t_\delta = \frac{1}{|b'(0, \alpha)|} \ln\left[\frac{x_0}{\delta}\right]. \tag{2.3}$$

2.1.2. *Marginal type*

The steady state is of the marginal type if $\alpha \in R^1$ and for a value $\alpha = \alpha_c$ we have

$$b(0, \alpha_c) = b'(0, \alpha_c) = 0, \quad b''(0, \alpha_c) \neq 0. \tag{2.4}$$

There exists a change of coordinates (refs. 5, 6) so that for α near α_c, x near the origin eq. (2.1) becomes

$$\dot{y} = y^2 - \beta(\alpha), \quad y(0) = y_0(x_0). \tag{2.5}$$

In (2.5), $\beta(\alpha)$ is a regular function of α and $\beta(\alpha_c) = 0$. We call $\alpha = \alpha_c$ the marginal bifurcation point. The flow of (2.5) is sketched in fig. 1. The bifurcation picture is shown in fig. 2. The marginal case was considered briefly by Kubo et al.[4]) and Nitzan et al.[7]).

Now suppose that $\beta > 0$ and $-\sqrt{\beta} < y_0 < \sqrt{\beta}$. One can calculate the time that it takes to reach a point y_1. We obtain (assuming $y_1 < y_0$)

$$t_{y_1} = \frac{1}{2\sqrt{\beta}} \left\{ \ln\left[\frac{y_1 - \sqrt{\beta}}{y_1 + \sqrt{\beta}}\right] - \ln\left[\frac{y_0 - \sqrt{\beta}}{y_0 + \sqrt{\beta}}\right] \right\}. \tag{2.6}$$

For small β we have

$$\frac{y_1 - \sqrt{\beta}}{y_1 + \sqrt{\beta}} = \frac{1 - \sqrt{\beta}/y_1}{1 + \sqrt{\beta}/y_1} = (1 - \sqrt{\beta}/y_1)^2 + \mathcal{O}(\beta). \tag{2.7}$$

Expanding the logarithms in (2.6) gives

$$t_{y_1} \sim \frac{1}{y_0} - \frac{1}{y_1} + \mathcal{O}(\beta) \tag{2.8}$$

so that t_{y_1} remains finite as $\beta \to 0$. Clearly, this result would not be obtained had we used the linearized version of (2.5):

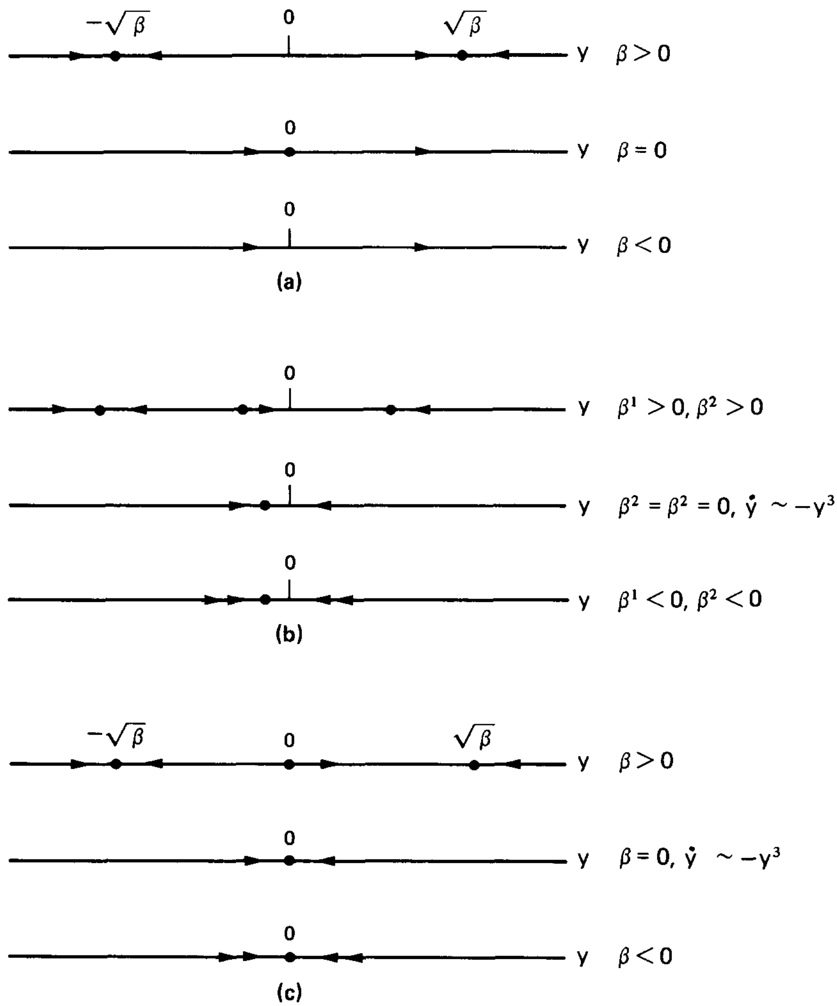

Fig. 1. Dynamics of the canonical systems. (a) Marginal type. (b) Critical type. (c) Hopf type.

$$\dot{y} = -2\sqrt{\beta}(y + \sqrt{\beta}). \tag{2.9}$$

In another possible situation $\beta = 0$. Suppose that $y_0 < 0$. The time to reach $\delta < 0$ from y_0 ($y_0 < \delta$) is (exactly)

$$t = -\frac{1}{\delta} + \frac{1}{y_0}. \tag{2.10}$$

The point of importance is that (2.8), (2.10) yield algebraic forms for the relaxation time, whereas (2.3) yields a logarithmic time (i.e., algebraic versus exponential relaxation).

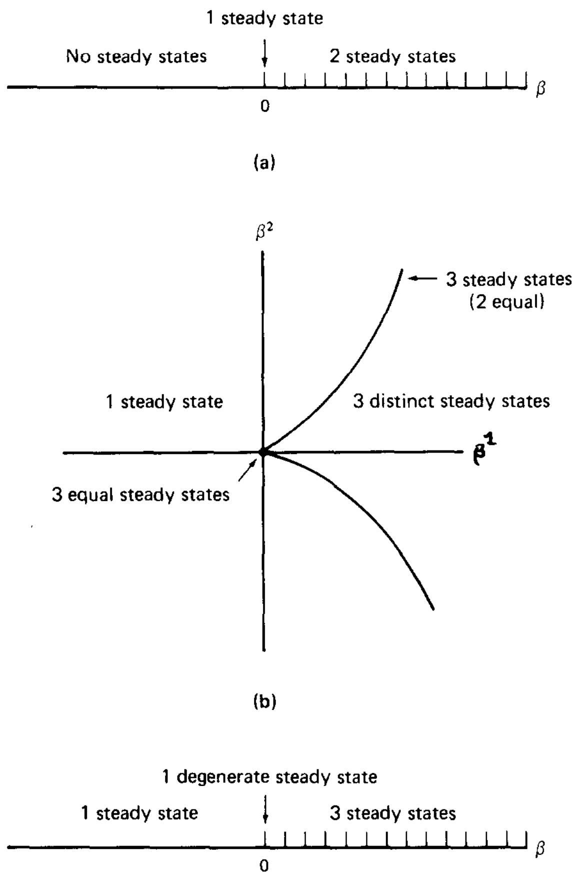

Fig. 2. Bifurcation pictures in parameter space. (a) Marginal type. (b) Critical type. (c) Hopf type.

2.1.3. *Critical type*

A steady state is of the critical type if $\alpha \in R^2$ and for $\alpha = \alpha_c$

$$b(0, \alpha_c) = b'(0, \alpha_c) = b''(0, \alpha_c) = 0, \quad b'''(0, \alpha_c) \neq 0. \tag{2.11}$$

The canonical form of the dynamics, for α near α_c and y near 0 is (for $b''' < 0$)

$$\dot{y} = -y^3 + \beta^1(\alpha)y + \beta^2(\alpha), \quad y(0) = y_0(x_0). \tag{2.12}$$

In (2.12), $\beta(\alpha)$ is a regular function of α and $\beta(\alpha_c) = 0$. We call $\alpha = \alpha_c$ the critical bifurcation point. The flow of (2.12) is sketched in fig. 1. The bifurcation picture is shown in fig. 2.

When $\alpha = \alpha_c$, we have

$$\dot{y} = -y^3 \tag{2.13}$$

so that the origin is very weakly attracting. The time that the system takes to reach $y = \delta$ from $y = y_0 > \delta$ is

$$t_\delta = -\tfrac{1}{2}\left[\frac{1}{y_0^2} - \frac{1}{\delta^2}\right]. \tag{2.14}$$

As in the marginal case, we obtain an algebraic, rather than exponential, decay rate.

2.1.4. *Hopf type*

A steady state is of the Hopf type if $\alpha \in R^1$ and when $\alpha = \alpha_c$

$$b(0, \alpha_c) = b'(0, \alpha_c) = b''(0, \alpha_c) = 0, \quad b'''(0, \alpha_c) \neq 0. \tag{2.15}$$

The canonical dynamics[8]) in this case (for $b''' < 0$) are

$$\dot{y} = -y(y^2 - \beta(\alpha)), \tag{2.16}$$

where $\beta = \beta(\alpha)$ is a regular function of α and $\beta(\alpha_c) = 0$. The flow of (2.16) is sketched in fig. 1. The bifurcation picture is sketched in fig. 2. It is important to note the difference between Hopf and critical cases (i.e. the number of parameters).

2.2. *Multi-dimensional theory: canonical forms*

We now consider

$$\dot{x} = b(x, \alpha), \quad x \in R^n, \quad \alpha \in R^1 \quad \text{or} \quad \alpha \in R^2, \tag{2.17}$$

with the origin a steady state. We let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of the matrix $B = (b^i_{,j})|_{x=0}$. For simplicity, we assume that there are n distinct eigenvalues and eigenvectors. Let k_+, k_-, k_0 denote the number of eigenvalues with real part positive, negative, and zero, respectively. The dynamical systems are classified as follows.

2.2.1. *Normal case*

In this case $k_0 = 0$. It is well known that (2.17) can be replaced by a change of variables $x \to y$ so that

$$\dot{y}^i = \lambda_i y^i + \mathcal{O}(y^2) \quad \text{and} \quad y^i(0) = y_0^i(x_0). \tag{2.18}$$

If $k_+ = 0$, then the origin is stable. If $k_+ > 0$, then R^n can be divided into two sub-spaces: an expanding part (W_e), and a contracting part (W_c) (fig. 3), with $\dim W_e + \dim W_c = n$.

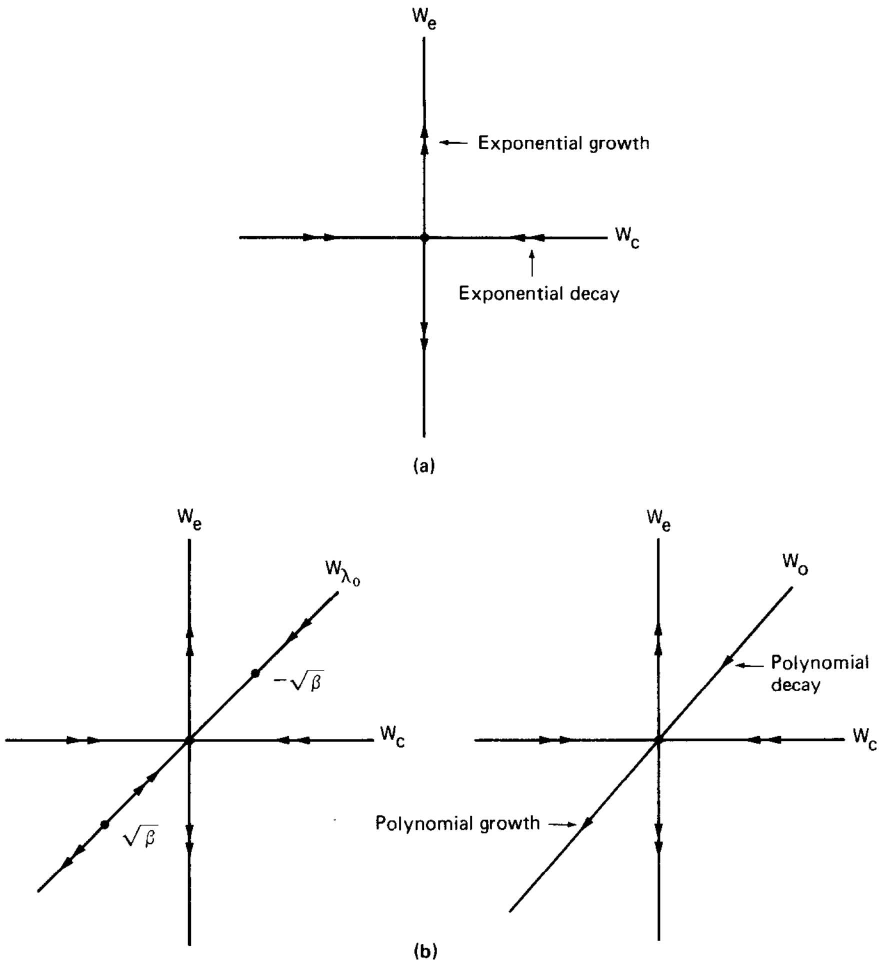

Fig. 3. Multidimensional phase spaces. (a) Normal type. (b) Marginal type: double arrows indicate exponential growth/decay; single arrows indicate polynomial growth/decay.

2.2.2. *Marginal case*

We now let $\alpha \in R^1$ vary. Then the eigenvalues of B are functions of α: $\lambda_k = \lambda_k(\alpha)$. When $\alpha = \alpha_c$ we assume that,

1) All eigenvalues are real. Exactly one eigenvalue $\lambda_0(\alpha_c) = 0$. There are k negative eigenvalues, $\lambda_1, \ldots, \lambda_k$ and $n-1-k$ positive eigenvalues $\lambda_{k+1}, \ldots, \lambda_{n-1}$.

2) There are enough eigenvectors. Let Z be a variable in the direction of the eigenvector corresponding to λ_0. Then from (2.17), we obtain,

$$\dot{Z} = \tilde{b}(y(Z), \alpha). \tag{2.19}$$

The marginal type steady state is characterized by

$$\tilde{b}(0, \alpha_c) = \tilde{b}_Z(0, \alpha_c) = 0, \quad \tilde{b}_{ZZ}(0, \alpha_c) \neq 0, \tag{2.20}$$

where a subscript indicates differentiation. In refs. 5 and 6, it is shown that (2.17) can be put into the form

$$\begin{aligned} \dot{y}^i &= \lambda_i y^i + \mathcal{O}(y^2), \quad y^i \in R^{n-1}, \\ \dot{y}^0 &= (y^0)^2 \pm \beta(\alpha) + \mathcal{O}(y^3), \quad y^0 \in R^1, \end{aligned} \tag{2.21}$$

Arnold[5]) and Shoshaitshvili[6]) show there exist transformations which eliminate the higher terms. When $\alpha = \alpha_c$, $\beta(\alpha_c) = 0$. The phase space R^n is now decomposed into a direct product

$$R^n = W_0 + W_e + W_c, \tag{2.22}$$

where W_0 is the manifold corresponding to λ_0 and W_e, W_c are the expanding and contracting sub-spaces, respectively. The assumption that all eigenvalues of B were real affected the form of the canonical equations. Complex eigenvalues are explicitly treated in the Hopf case.

2.2.3. *Critical type*

In this case, $\alpha \in R^2$. The eigenvalues of B are still denoted by $\lambda(\alpha)$. We assume:

1) When $\alpha = \alpha_c$ there is one zero eigenvalue, λ_0, k negative eigenvalues and $n - k - 1$ positive eigenvalues. All eigenvalues are real.

2) There are enough eigenvectors. Let Z be in the direction of the eigenvector belonging to $\lambda_0(\alpha)$. Then from (2.17), we obtain

$$\dot{Z} = \tilde{b}(y(Z), \alpha). \tag{2.23}$$

The critical type steady state is characterized by

$$\tilde{b}(0, \alpha_c) = \tilde{b}_Z(0, \alpha_c) = \tilde{b}_{ZZ}(0, \alpha_c) = 0, \quad \tilde{b}_{ZZZ}(0, \alpha_c) \neq 0. \tag{2.24}$$

In refs. 5 and 6, it is shown that the canonical dynamics are

$$\begin{aligned} \dot{y}^i &= \lambda_i y^i + \mathcal{O}(y^2), \\ \dot{Z} &= \pm(Z^3) - \beta_1(\alpha) Z - \beta_2(\alpha) + \mathcal{O}(Z^4). \end{aligned} \tag{2.25}$$

In (2.25), $\beta(\alpha)$ is a regular function that vanishes when $\alpha = \alpha_c$. The ($\pm$) sign in (2.25) corresponds to the sign of $\tilde{b}_{ZZZ}(0, \alpha_c)$. The decomposition of the phase space R^n is sketched in fig. 4.

2.2.4. *Hopf type*

In the Hopf case, $\alpha \in R^1$ and some of the eigenvalues are complex. When

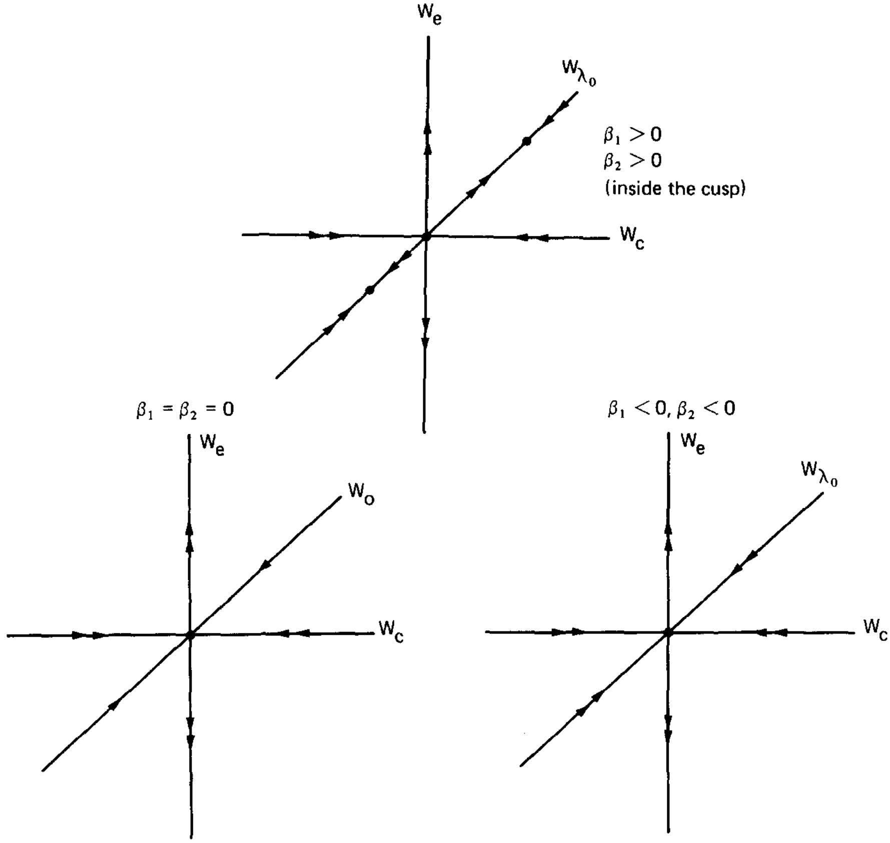

Fig. 4. Decomposition of phase space in the critical case. Double arrows indicate exponential growth/decay; single arrows indicate polynomial growth/decay.

$\alpha = \alpha_c$ one eigenvalue, $\lambda_0(\alpha)$ is a pure imaginary with

$$\frac{d}{d\alpha} \operatorname{Re} \lambda_0(\alpha)|_{\alpha_c} \neq 0. \tag{2.26}$$

Thus, as α crosses α_c, a pair of eigenvalues crosses from the left half plane into the right half plane.

Let $x = r\,e^{i\theta}$. Fenichel[8]) (also see Arnold[5])) has shown that the canonical dynamics are

$$\dot{r} = \pm(b_1 r^3 - \eta\gamma_1 r), \tag{2.27}$$

$$\dot{\theta} = \lambda_2 + b_2 r^2 + \eta\gamma_1 r, \tag{2.28}$$

where γ_1 is

$$\gamma_1 = \frac{d}{d\alpha} \operatorname{Re} \lambda_0(\alpha)|_{\alpha_c},$$

$\lambda_2 > 0$ and $b_1, b_2 \neq 0$. The function $\eta = \eta(\alpha)$ is regular and $\eta(0) = 0$.

2.2.5. *Relaxation rates*

Given an initial displacement from the origin

$$y(0) = \{y_0, \ldots, y_{n-1}\} \tag{2.29}$$

it is clear that the appropriate relaxation (or growth) rate of the kth component (or mode) can be explicitly calculated by using the canonical forms. The calculations reveal exponential growth in W_e, decay in W_c, and polynomial growth or decay in W_0 (at bifurcation points). Thus, we have a complete, albeit local, deterministic theory for relaxation phenomena in the vicinity of critical points.

3. Stochastic theory of relaxation: formulation and one-dimensional results

The deterministic theory of section 2 is approximate in that it ignores fluctuations. Since the deterministic dynamics vanish at a steady state, the proper theory of relaxation phenomena is a stochastic one. The theory given here is still phenomenological, but it may be possible to connect it to underlying statistical physics.

3.1. *Stochastic kinetic equation and diffusion approximation*

We replace the deterministic kinetic equation (2.17) by the Langevin equation:

$$\frac{d\tilde{x}^i_\tau}{dt} = b^i(\tilde{x}_\tau) + \frac{\sqrt{\epsilon}}{\tau}\sigma^i_j(x)\tilde{y}^j(t/\tau^2). \tag{3.1}$$

In (3.1), τ is a small parameter relating the time scales of the fluctuations and the deterministic dynamics, ϵ is a small parameter characterizing the intensity of the fluctuations. The process $\tilde{y}^j$ is a zero mean, mixing process (for more exact assumptions, see ref. 9). The field $\sigma^i_j(x)$ is assumed to be known, or given by some prescription.

The process $\tilde{y}(s)$ in (3.1) has correlations. Hence, this model is more general than "white noise" models. We let

$$\gamma^{k\ell} = \int_0^\infty E[\tilde{y}^k(s)\tilde{y}^\ell(0)]\, ds. \tag{3.2}$$

As $\tau \to 0$, $\tilde{x}_\tau \to \tilde{x}$, a diffusion process[9]). If $u_0(x)$ is a bounded, measurable function and

$$u(x, t) = E\{u_0(\tilde{x}(t)) \mid \tilde{x}(0) = x\}, \tag{3.3}$$

then $u(x, t)$ satisfies the backward equation

$$u_t = \frac{\epsilon a^{ij}}{2} u_{ij} + (b^i + \epsilon c^i)u_i. \tag{3.4}$$

In (3.4), subscripts indicate partial differentials and repeated indices are summed from 1 to n. The coefficients a^{ij}, c^i are

$$a^{ij} = \sigma^i_k(x)\sigma^j_\ell(x)[\gamma^{k\ell} + \gamma^{\ell k}], \tag{3.5}$$

$$c^i = \gamma^{k\ell}\sigma^j_k(x)\frac{\partial}{\partial x^j}\sigma^i_\ell(x). \tag{3.6}$$

In practice $a(x)$ and $c(x)$ cannot be calculated from first principles, but some prescription must be given for their calculation (e.g., ref. 10).

Let N be a neighborhood of a stable steady state or, more generally, a domain in R^n. We set

$$\begin{aligned} &u(x, t) = 1 \qquad x \in N \\ &u(x, t) \to 0 \qquad \text{as distance from } x \text{ to } N \to \infty \\ &u(x, 0) = \begin{cases} 0 & x \notin N \\ 1 & x \in N. \end{cases} \end{aligned} \tag{3.7}$$

Then, $u(x, t)$ is the probability that $\tilde{x}(t)$ has entered N by time t, given that $\tilde{x}(0) = x$.

For stochastic relaxation theory, we are interested in the expected time to enter N, given $\tilde{x}(0) = x$ and that the process enters N:

$$T(x) = \int_0^\infty t u_t(x, t)\, \mathrm{d}t. \tag{3.8}$$

It can be shown that $T(x)$ satisfies[13])

$$\frac{\epsilon a^{ij}}{2} T_{ij} + (b^i + \epsilon c^i)T_i = -\bar{u}(x), \tag{3.9}$$

where

$$\bar{u}(x) = \lim_{t\to\infty} u(x, t). \tag{3.10}$$

Namely, $\bar{u}(x)$ is the probability that the process eventually enters N, given that $\tilde{x}(0) = x$. The boundary conditions appropriate to (3.9) are

$$T(x) = 0, \quad x \in N \tag{3.11}$$

and a growth condition as distance $(x, N) \to \infty$.

3.2. *Exact solution and canonical forms*

When $x \in R^1$, eq. (3.9) is an ordinary differential equation

$$\frac{\epsilon a}{2} T_{xx} + (b(x) + \epsilon c(x))T_x = -\bar{u}(x). \tag{3.12}$$

Let $N = \{\hat{x}\}$. Then the solution of (3.12) is

$$T(x) = \int_x^{\hat{x}} \frac{2}{\epsilon a} \exp\left[-\frac{2}{\epsilon} \int^s \frac{b + \epsilon c}{a} \mathrm{d}y\right] \int_{-\infty}^{s} \bar{u}(x') \exp\left[\frac{2}{\epsilon} \int^{x'} \frac{b + \epsilon c}{a} \mathrm{d}y\right] \mathrm{d}x' \, \mathrm{d}s. \tag{3.13}$$

Eq. (3.13) has a rather complicated asymptotic analysis. Instead of studying the asymptotic analysis of (3.13), we return to (3.12) and set, for convenience $a \equiv 2$, $\bar{u}(x) \equiv 1$, and $c = 0$. We will analyze (3.12) and obtain certain special functions. These functions will be generalized in section 4, for the solution of multi-dimensional problems. Our analysis is based on matched asymptotic expansions (e.g. ref. 11).

Away from the zeros of $b(x)$, we set $\epsilon = 0$, so that (3.12) becomes

$$b(x)T_x = -1. \tag{3.14}$$

This is the "outer" equation.

Near the zeros of $b(x)$, (3.14) breaks down. We need to stretch coordinates in (3.12) to obtain the appropriate "inner" equations. We shall analyze (3.12) by using the canonical form of $b(x)$.

3.2.1. *Normal case*

In the normal case, $b(x) = \pm x$, with (+) indicating that the origin is an unstable steady state, (−) indicating a stable steady state. Introducing $z = x/\sqrt{\epsilon}$, the inner equation becomes

$$T_{zz} \pm zT_z = -1. \tag{3.15}$$

3.2.2. *Marginal case*

In the marginal case, the canonical dynamics are $b(x) = \pm(x^2) - \tilde{\alpha}$. We introduce the stretched variables $z = x/\epsilon^{1/3}$ and $\alpha = \tilde{\alpha}/\epsilon^{2/3}$, so that the inner equation is

$$T_{zz} \pm (z^2 - \alpha)T_z = -1/\epsilon^{1/3}. \tag{3.16}$$

3.2.3. *Critical case*

In the critical case, the canonical dynamics are $b(x) = \pm x^3 + \tilde{\beta}_1 x + \tilde{\beta}_2$. We introduce stretched variables $z = x/\epsilon^{1/4}$, $\beta_1 = \tilde{\beta}_1/\epsilon^{1/2}$ and $\beta_2 = \tilde{\beta}_2/\epsilon^{3/4}$ and obtain the inner equation

$$T_{zz} + (\pm z^3 + \beta_1 z + \beta_2)T_z = -1/\epsilon^{1/2}. \tag{3.17}$$

3.2.4. *Hopf case*

In the Hopf case, the canonical dynamics are $b(x) = -x^3 + \tilde{\beta}x$. We introduce the stretched variables $z = x/\epsilon^{1/4}$, $\beta = \tilde{\beta}/\epsilon^{1/2}$ and obtain the inner equation

$$T_{zz} + (-z^3 + \beta z)T_z = -1/\epsilon^{1/2}. \tag{3.18}$$

Eqs. (3.15)–(3.18) define certain incomplete special functions. These special functions will be used in the next section to construct asymptotic solutions of multi-dimensional problems.

4. Stochastic theory: asymptotic results

When $x \in R^n$, $n \geq 2$, eq. (3.9) will usually not have exact solutions. Consequently, approximate techniques are required. The methods used here are closely related to those in ref. 10. The basic idea is to generalize the one-dimensional inner solutions; we call the method a generalized ray method. Although the normal case does not represent a "critical" point, we include it for completeness.

4.1. *Normal case*

We suppose that the origin is a simple steady state (fig. 5) and that it is stable. We seek a solution of (3.9) in the form

$$T(x) = g(x)F(\psi/\sqrt{\epsilon}) + h(x)\epsilon^{1/2}F'(\psi/\sqrt{\epsilon}) + k(x). \tag{4.1}$$

In eq. (4.1), $F(z)$ is a special function satisfying

$$\frac{d^2F}{dz^2} = z\frac{dF}{dz} - 1 \tag{4.2}$$

and the functions $\psi(x)$, $g(x)$, $h(x)$, and $k(x)$ are to be determined. In order to completely analyze the problem, we assume that g, h, k have expansions

$$g(x) = \Sigma\, g^n(x)\epsilon^n, \quad h(x) = \sum h^n(x)\epsilon^n \quad \text{and} \quad k(x) = \sum k^n(x)\epsilon^n. \tag{4.3}$$

Consequently, the construction given here represents the first term in the asymptotic solution of (3.9).

When derivatives are evaluated, (4.2) is used to replace $F''(\psi/\sqrt{\epsilon})$ by $(\psi/\sqrt{\epsilon})F'(\psi/\sqrt{\epsilon}) - 1$. Then terms are collected according to powers of ϵ. We

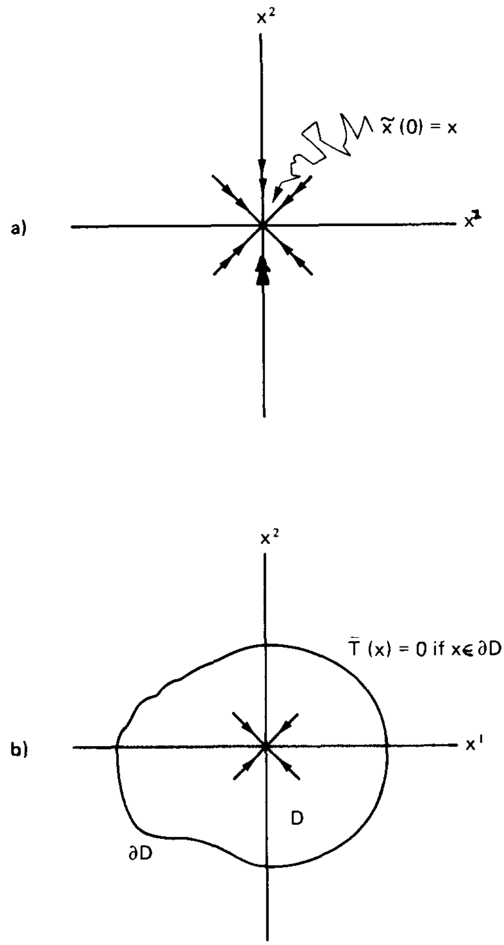

Fig. 5. Stochastic relaxation problems in the normal case. (a) The relaxation problem. (b) The first exit problem.

obtain:

$$\begin{aligned}
-\bar{u}(x) &= \epsilon^{-1/2}\left[b^i\psi_i + \frac{a^{ij}}{2}\psi_i\psi_j\psi\right](g + h\psi)F' \\
&\quad + \epsilon^0(b^i g_i)F + \epsilon^0\left(b^i k_i + \frac{a^{ij}}{2}\psi_i\psi_j g\right) \\
&\quad + \epsilon^{1/2}F'\left[b^i h_i + \frac{a^{ij}}{2}g\psi_j + a^{ij}h_i\psi_j\psi + \frac{a^{ij}}{2}h\psi_{ij}\psi\right. \\
&\quad \left. + \frac{a^{ij}}{2}\psi_i\psi_j h - gc^i\psi_i + hc^i\psi_i\psi\right].
\end{aligned} \tag{4.4}$$

The leading terms vanish if

$$b^i\psi_i + \frac{a^{ij}}{2}\psi_i\psi_j\psi = 0, \tag{4.5}$$

$$b^i g_i = 0, \tag{4.6}$$

$$b^i k_i + \frac{a^{ij}}{2}\psi_i\psi_j g = -\bar{u}(x). \tag{4.7}$$

First consider (4.5). Since $b^i(0) = 0$ for all i, we set $\psi(0) = 0$, in order to keep $\psi(x)$ regular. Then (4.5) can be solved by the method of characteristics. We note that the transformation $\Phi = \frac{1}{2}\psi^2$ converts (4.5) to

$$b^i\Phi_i + \frac{a^{ij}}{2}\Phi_i\Phi_j = 0, \tag{4.8}$$

which is a Hamiltonian–Jacobi equation (also see ref. 12). Then, we can solve the Hamilton–Jacobi equation in terms of characteristics:

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_l = -\frac{\partial H}{\partial x^i}, \quad \dot{\Phi} = \tfrac{1}{2}a^{ij}p_ip_j, \tag{4.9}$$

where

$$H(x, p) = b^i p_i + \frac{a^{ij}}{2}p_ip_j. \tag{4.10}$$

Starting at the origin, the phase plane is covered with trajectories, called rays, along which ψ (or Φ) is calculated. Thus, the value of ψ at any point x is known.

Eq. (4.3) indicates that g is constant on deterministic trajectories. Since all trajectories intersect at the origin, g must have the same value on all trajectories. At the origin, (4.7) becomes

$$\frac{a^{ij}}{2}\psi_i\psi_j g = -\bar{u}(0). \tag{4.11}$$

Thus

$$g = \frac{-\bar{u}(0)}{(a^{ij}/2)\psi_i\psi_j}. \tag{4.12}$$

We set $k(0) = 0$ as initial data for (4.7).

If we set $F(0) = F'(0) = 0$ as initial conditions in (4.2), then the leading term of the asymptotic solution satisfies $T(0) \equiv 0$.

The $\mathcal{O}(\epsilon^{1/2}F')$ term in (4.4) vanishes if

$$b^ih_i + \frac{a^{ij}}{2}g\psi_{ij} + a^{ij}h_i\psi_j\psi + \frac{a^{ij}}{2}h\psi_{ij}\psi + \frac{a^{ij}}{2}\psi_i\psi_jh - gc^i\psi_i + hc^i\psi_i\psi = 0. \tag{4.13}$$

At the origin $b^i(0) = \psi(0) = 0$, so that (4.13) becomes

$$h(0) = \frac{1}{(a^{ij}/2)\psi_i\psi_j}\left\{\frac{-a^{ij}}{2}\psi_{ij}g + c^i\psi_i g\right\}. \tag{4.14}$$

Eq. (4.13) can be solved by the method of characteristics, with initial data given by (4.14).

Thus, we have completely constructed the leading term of the asymptotic solution of (3.9).

As a by-product of our method, we are able to approximately solve the Kolmogorov first exit problem, recently considered by Matkowsky and Schuss[13]) using matched asymptotic expansions. This problem is the following: suppose that the origin is surrounded by a domain D, with boundary ∂D. Find the expected time that the process takes to hit the boundary (i.e. the mean exit time from D (fig. 5b)) from x.

We follow the arguments leading to eqs. (4.1)–(4.11), except that the initial data for F, F' and $k(x)$ change. We set $k(x) \equiv 0$ on ∂D. We distinguish two cases:

i) The boundary ∂D is a contour of ψ (or Φ) say, $\psi = \psi_D$ on ∂D. We set

$$F(\psi_D/\sqrt{\epsilon}) = F'(\psi_D/\sqrt{\epsilon}) = 0 \tag{4.15}$$

when solving (4.2). In this case, T vanishes identically on ∂D.

ii) The boundary ∂D is not a contour of ψ. Let ψ_{I} and ψ_{II} denote the maximum and minimum values of ψ on ∂D. Then $T \not\equiv 0$ on ∂D, but it can be shown that on ∂D

$$|T| \leq |\ln(\psi_{\mathrm{I}}/\psi_{\mathrm{II}})| + \text{exponentially small terms}. \tag{4.16}$$

Hence, if $|\ln(\psi_{\mathrm{I}}/(\psi_{\mathrm{II}})|$ is small, then $|T(x)|$ will be small on the boundary.

4.2. *Marginal case*

In some senses, the marginal case has the least interesting dynamics. The dynamical problem we consider here is sketched in fig. 6. When the deterministic system has two nodes (Q_0, Q_1) and one saddle (S), even if the process starts near Q_0, it will eventually reach Q_1, due to the proximity of Q_0 and S. The proper question in the stochastic theory involves the time to cross some given curve R. We note that such a time may be infinite in the deterministic case. We seek a solution of (3.9) of the form

$$T(x) = g(x)B(\psi/\epsilon^{1/3}, \beta/\epsilon^{2/3}, 1/\epsilon^{1/3}, \gamma_2) + h(x)\epsilon^{2/3}B'(\psi/\epsilon^{1/3}, \beta/\epsilon^{2/3}, 1/\epsilon^{1/3}, \gamma_2) + k(x). \tag{4.17}$$

In (4.17), $B(z, \alpha, \lambda_1, \lambda_2)$ satisfies

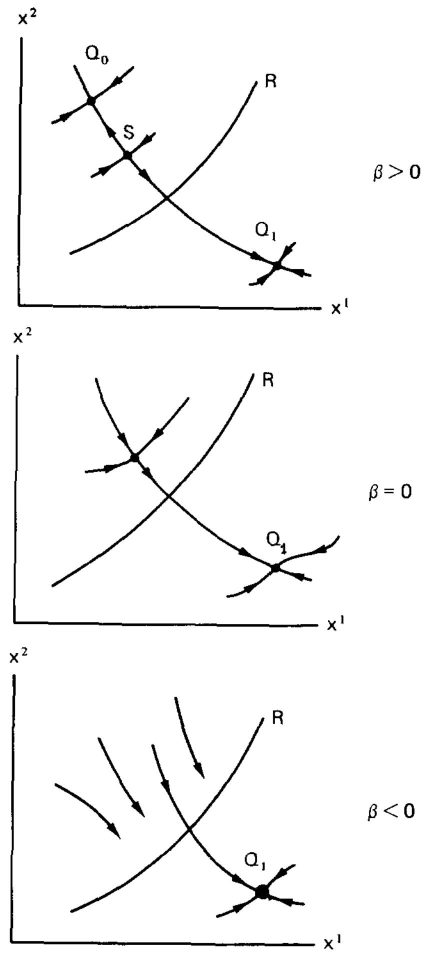

Fig. 6. Relaxation problems in the marginal case.

$$\frac{d^2B}{dz^2} = -(z^2 - \alpha)\frac{dB}{dz} - \lambda_1 + \lambda_2 z \tag{4.18}$$

and $g(x)$, $h(x)$, $k(x)$, $\psi(x)$ and the parameters α, γ_2 are to be determined. We proceed as in section 4.1. Instead of eqs. (4.5)–(4.7) we obtain

$$b^i\psi_i - \frac{a^{ij}}{2}\psi_i\psi_j(\psi^2 - \beta_0) = 0, \tag{4.19}$$

$$b^i g_i = 0, \tag{4.20}$$

$$b^i k_i - \frac{a^{ij}}{2}\psi_i\psi_j g(1 - \gamma_2\psi) = -\bar{u}(x). \tag{4.21}$$

In (4.17), we have set $\beta = \Sigma\, \beta_k \epsilon^k$.

We set $\psi^2 = \beta_0$ at Q_0 and at S. In particular $\psi(Q_0) = +\sqrt{\beta_0}$ and $\psi(S) = -\sqrt{\beta_0}$. The value of β_0 can be determined by an iterative procedure[10]. We pick an initial value of $\beta_0 = \beta_0^{(0)}$ and solve (4.19) by the method of characteristics, starting at Q_0, where $\psi = \sqrt{\beta_0^{(0)}}$. Some rays will approach S. As a ray approaches S, ψ should approach $-\sqrt{\beta_0^{(0)}}$. If it does not, then the $\beta_0^{(0)}$ must be replaced by a second iterate $\beta_0^{(1)}$. The method of false position can be used to calculate iterates of β_0. This procedure can be repeated until β_0 is known to any desied accuracy. (In ref. 10 a discussion of this calculation is given in more detail.)

Eq. (4.20) indicates that g is a constant. At Q_0 and S, which are denoted generically by P, we have, from (4.21):

$$-\frac{a^{ij}}{2}\,\psi_i\psi_j|_{\mathrm{P}}\, g(1-\gamma_2\psi(\mathrm{P})) = -\bar{u}(\mathrm{P}). \tag{4.23}$$

These are two equations for the unknowns g and γ_2. We set $k=0$ on R and assume that R is a level curve of ψ, with $\psi = \psi_R$ on R. Then we set

$$B(\psi_R/\epsilon^{1/3}, \beta, 1/\epsilon^{1/3}, \gamma_2) = B'(\psi_R/\epsilon^{1/3}, \beta, 1/\epsilon^{1/3}, \gamma_2) = 0. \tag{4.21}$$

With these choices, $T(x) \equiv 0$ if $x \in R$.

At the bifurcation point $\eta = 0$ (the marginal bifurcation) Q_0 and S coalesce. Then $\beta_0 \equiv 0$, and it can be shown that $\gamma_2 \equiv 0$ [10]. At the saddle-node Q_0/S, eq. (4.23) still provides one equation for g:

$$g = \frac{\bar{u}(\mathrm{P})}{(a^{ij}/2)\psi_i\psi_j}. \tag{4.24}$$

Elsewhere, it is shown that all these constructions are regular at the bifurcation point[10].

In section 5, we consider an example of a chemical system exhibiting the marginal bifurcation.

4.3. *Critical case*

Now consider a system with three steady states, P_0, P_1, and P_2 when $\alpha_1, \alpha_2 > 0$. When $\alpha_1 = \alpha_2 = 0$ the three steady states coalesce into a critical type steady state. When $\alpha_1, \alpha_2 < 0$ there is only one real steady state; it is assumed to be stable. If $\alpha_1, \alpha_2 > 0$, we surround P_2 by a domain N and pose the following stochastic relaxation problem: Find the expected time to enter N, given the initial position. Clearly there is an analogous problem for a neighborhood N of P_0. When there is only one steady state P, we surround P by N. We note that if N shrinks to P, then we have the expected time to "reach" P, conditioned on initial position. We also note that $T(x) \equiv 0$ if $x \in N$.

We seek a solution of (3.9) in the form

$$T(x) = g(x)Q(\psi/\epsilon^{1/4}, \alpha/\epsilon^{1/2}, \beta/\epsilon^{3/4}, 1/\epsilon^{1/2}, \gamma_1/\epsilon^{1/4}, \gamma_2) \\ + h(x)\epsilon^{3/4}Q'(\psi/\epsilon^{1/4}, \alpha/\epsilon^{1/2}, \beta/\epsilon^{3/4}, 1/\epsilon^{1/2}, \gamma_1/\epsilon^{1/4}, \gamma_2) + k(x), \tag{4.25}$$

Where $Q(z, \alpha, \beta, \gamma_1, \gamma_2, \gamma_3)$ satisfies

$$\frac{d^2Q}{dz^2} = \pm(z^3 - \alpha z - \beta)\frac{dQ}{dZ} - \gamma_1 + \gamma_2 z + \gamma_3 z^2. \tag{4.26}$$

The (+) sign in (4.26) corresponds to the steady state P being stable, the (−) sign to it being unstable. We consider the case in which P is stable.

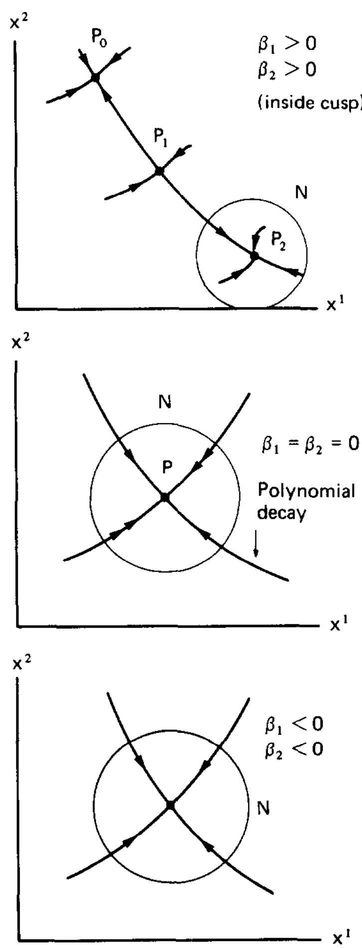

Fig. 7. Relaxation problems in the critical case.

Instead of (4.5), we obtain

$$b^i\psi_i + \frac{a^{ij}}{2}\psi_i\psi_j(\psi^3 - \alpha\psi - \beta) = 0. \tag{4.27}$$

When there are three steady states, α and β are determined by a procedure analogous to the one in section 4.2. Namely, at the steady states we set

$$\psi^3 - \alpha\psi - \beta = 0. \tag{4.28}$$

The method of characteristics is then used to determine α and β by an iterative procedure. When the three steady states coalesce $\alpha = \beta = 0$. When there is one real and two imaginary steady states, then $\alpha, \beta < 0$ and can be determined by power series. Such series are constructed elsewhere[10]).

Instead of (4.7), we obtain

$$b^i k_i + \frac{a^{ij}}{2}\psi_i\psi_j g(-1 + \gamma_2\psi + \gamma_3\psi^2) = -\bar{u}(x). \tag{4.29}$$

At the steady states, we obtain

$$\frac{a^{ij}}{2}\psi_i\psi_j g(-1 + \gamma_2\psi + \gamma_3\psi^2) = -\bar{u}(x). \tag{4.30}$$

When there are three real steady states, we obtain three equations for g, γ_2, and γ_3. When two steady states coalesce, it can be shown that $\gamma_3 = 0$. We still have two equations for g and γ_2. Finally, when all three coalesce, $\gamma_2 = \gamma_3 = 0$ and (4.30) becomes one equation for g.

We obtain an equation for $h(x)$ that is analogous to (4.13), and is treated in an analogous fashion. The initial values of Q and Q' in (4.26) are determined so that $T(x) \equiv 0$ if $x \in \partial N$.

4.4. *Hopf case*

The Hopf type dynamical system is treated in an identical fashion to the marginal and critical type systems. We seek a solution of (3.9) in the form

$$\begin{aligned} T(x) = {} & g(x)H(\psi/\epsilon^{1/4}, \beta/\epsilon^{1/2}, 1/\epsilon^{1/2}, \gamma_2/\epsilon^{1/4}) \\ & + \epsilon^{1/4}H'(\psi/\epsilon^{1/4}, \beta/\epsilon^{1/2}, 1/\epsilon^{1/2}, \gamma_2/\epsilon^{1/4})h(x) + k(x), \end{aligned} \tag{4.31}$$

where $H(z, \beta, \gamma_1, \gamma_2)$ satisfies

$$\frac{d^2H}{dz^2} = \pm(z^3 - \beta z)\frac{dH}{dz} - \gamma_1 + \gamma_2 z. \tag{4.32}$$

The (+) sign corresponds to a stable limit cycle and unstable focus, the (−) sign corresponds to an unstable limit cycle and stable focus. The analysis proceeds exactly as in sections 4.2 and 4.3.

5. Substrate inhibited reactions: a marginal type steady state

The following equations model a substrate inhibited chemical reaction in an open reactor[10,14]):

$$\dot{x}^1 = \frac{-1.4x^1}{1.5 + x^1 + 13(x^1)^2} - 0.069979x^1 + 0.25901 - \frac{-x^1x^2}{1 + 10x^1x^2}, \tag{5.1}$$

$$\dot{x}^2 = 0.09 - \frac{x^1x^2}{1 + 10x^1x^2}, \tag{5.2}$$

where x^1 and x^2 are dimensionless "concentration" variables. The steady state (0.4359, 2.065) is a saddle node, it is a marginal type steady state. The steady state (1.46, 0.52) is a stable node. The phase portrait is shown in fig. 8, along with a first exit boundary. The theory of section 4.2 applies. We wish to calculate the expected time to hit R, conditioned on initial position. Using the birth and death approach to chemical kinetics[15]), ϵa can be modeled as[10]):

$$\epsilon a = \epsilon \begin{pmatrix} (\lambda_1 + \mu_1)x^1 & \dfrac{x^1x^2}{1 + 10x^1x^2} \\ \dfrac{x^1x^2}{1 + 10x^1x^2} & (\lambda_2 + \mu_2)x^2 \end{pmatrix}, \tag{5.3}$$

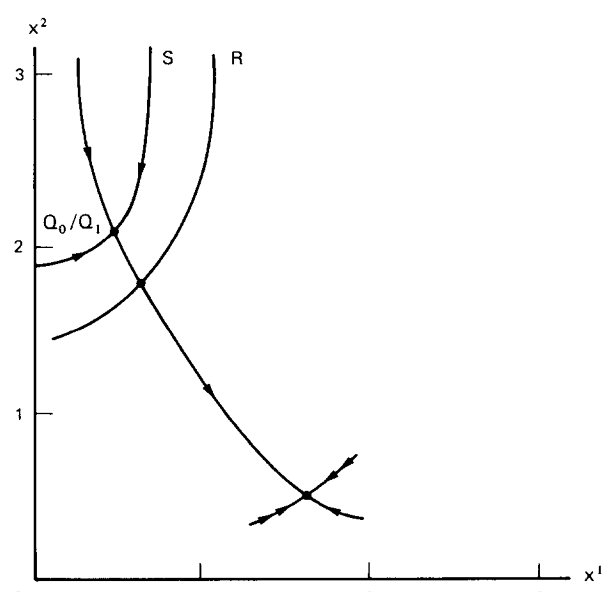

Fig. 8. Deterministic phase portrait at the marginal bifurcation. The boundary R was used in the calculation of the mean exit time.

TABLE I
Comparison of the theory and Monte-Carlo experiments in the marginal bifurcation

Test point	$T(x)$ theory	$T(x)$ experiment (# trials)
(0.42, 2.06)	60.3	56.4 (950)
(0.38, 2.36)	104.1	91.2 (400)
(0.20, 2.0)	66.1	62.4 (2000)
(0.3, 1.8)	37.7	35.0 (1550)
(0.16, 2.4)	119.6	103.5 (400)
(0.7, 2.2)	36.1	31.4 (1750)
(0.6, 2.4)	74.9	68.2 (800)

where

$$(\lambda_1 + \mu_1)x^1 = \frac{1.4x^1}{1.5 + x^1 + 13(x^1)^2} + 0.069979x^1 + 0.25901 + \frac{x^1x^2}{1 + 10x^1x^2}, \tag{5.4}$$

$$(\lambda_2 + \mu_2)x^2 = 0.09 + \frac{x^1x^2}{1 + 10x^1x^2}. \tag{5.5}$$

The parameter ϵ characterizes the intensity of the fluctuations. In table I, we compare the theory of section 4 with Monte-Carlo experiments, for $\epsilon = 0.01$.

6. Kinetic model of the ferromagnet

We shall give an analysis of the mean field ferromagnet, similar to that of Griffiths et al.[2]). The problem is one dimensional, so that the full theory of section 4 is not needed. However, this application illustrates many of the ideas that run throughout this paper.

Consider N spins, with $\sigma_i = \pm 1$, in a magnetic field H. Let J be a coupling constant. The Hamiltonian is

$$\tilde{H} = \frac{-J}{N} \sum \sigma_i\sigma_j - \mu H \sum \sigma_i - (1/2)J. \tag{6.1}$$

Let

$$n = \tfrac{1}{2}\left\{N + \sum \sigma_i\right), \tag{6.2}$$

denote the number of spins "pointing up." Then (6.1) becomes

$$\tilde{H} \equiv \Phi(n) = \frac{-J(2n - N)^2}{2N} - \mu H(2n - N). \tag{6.3}$$

A mean field approach is used; assume that the number of spins pointing up is really a statistical variable, $\tilde{n}(t)$. The statistical behavior of $\tilde{n}(t)$ is described by transition probabilities[2]):

$$\Pr\{\tilde{n}(\tau+\delta\tau)-\tilde{n}(\tau)=1 \mid \tilde{n}(\tau)=n\}$$
$$=\frac{N-n}{N}\exp\left[\frac{-\beta}{2}(\Phi(n+1)-\Phi(n))\right]\delta\tau+o(\delta\tau), \quad (6.4)$$

$$\Pr\{\tilde{n}(\tau+\delta\tau)-\tilde{n}(\tau)=-1 \mid \tilde{n}(\tau)=n\}$$
$$=\frac{n}{N}\exp\left[\frac{-\beta}{2}(\Phi(n-1)-\Phi(n))\right]\delta\tau+o(\delta\tau), \quad (6.5)$$

where $\beta = 1/k_B T$. Assume that the probability of all other transitions is $o(\delta\tau)$. In deriving (6.4–5), we have restated the argument in ref. 2. We follow ref. 2 and introduce a "continuous" variable

$$\tilde{x}(t)=\frac{\sum \sigma_i}{N}=\frac{2\tilde{n}-N}{N}. \quad (6.6)$$

If $\delta\tilde{x}=\tilde{x}(\tau+\delta\tau)-\tilde{x}(\tau)$, then (6.4–5) become

$$\Pr\{\delta\tilde{x}=2/N \mid \tilde{x}(\tau)=x\}=\frac{1-x}{2}\exp\left\{-\beta\left(-xJ-\frac{J}{N}-H\mu\right)\right\}\delta\tau+o(\delta\tau), \quad (6.7)$$

$$\Pr\{\delta\tilde{x}=-2/N \mid \tilde{x}(\tau)=x\}=\frac{1+x}{2}\exp\left\{-\beta\left(xJ-\frac{J}{N}+H\mu\right)\right\}\delta\tau+o(\delta\tau). \quad (6.8)$$

We set $\alpha = J\beta$, $\delta = \beta\mu H$ and introduce a macroscopic "physical" time defined by

$$t\equiv\frac{\tau}{N}. \quad (6.9)$$

Thus, we construct drift and diffusion coefficients of the form

$$b(x)=\lim_{\delta t\to 0}\frac{1}{\delta t}E\{\delta\tilde{x} \mid \tilde{x}(t)=x\} \quad (6.10)$$
$$=(1-x)\exp\left[\alpha x+\frac{\alpha}{N}+\delta\right]-(1+x)\exp\left[-\alpha x+\frac{\alpha}{N}-\delta\right] \quad (6.11)$$

and

$$a(x)=\lim_{\delta t\to 0}\frac{1}{\delta t}E\{(\delta\tilde{x})^2 \mid \tilde{x}(t)=x\} \quad (6.12)$$
$$=\frac{1}{N}\left\{(1-x)\exp\left(\alpha x+\frac{\alpha}{N}+\delta\right)+(1+x)\exp\left(-\alpha x+\frac{\alpha}{N}-\delta\right)\right\}. \quad (6.13)$$

The average value of $\tilde{x}(t)$ evolves according to

$$\dot{x} = b(x, \alpha, \delta) = 2\,\mathrm{e}^{\alpha/N}\{\sinh(\alpha x + \delta) - x\cosh(\alpha x + \delta)\}, \tag{6.14}$$

subject to $-1 \le x \le 1$. The steady states and true (physical) equilibrium are solutions of $b(x, \alpha, \delta) = 0$. Therefore, one obtains steady states as the solution of

$$x = \tanh(\alpha x + \delta). \tag{6.15}$$

Eq. (6.15) is usually obtained by a statistical thermodynamics argument (e.g. 16, p. 101).

This agreement adds support to the stochastic approach. Not only does the stochastic approach yield the equilibrium solution, it gives dynamics and the steady states. As is well known, eq. (6.15) may have 1, 2, or 3 solutions, depending upon the values of α and δ. In fig. 9a, b we illustrate the graphical solution of (6.15) for zero field ($\delta = 0$). When $\delta = 0$, x_0 and x_2 are both thermodynamically, and kinetically, stable. However, for $\delta \neq 0$, one of x_0, x_2 becomes kinetically stable (thermodynamically metastable) while the other is the true thermodynamic (and kinetic) equilibrium (fig. 9c). The kinetic condition of criticality is that, when $\delta = 0$

$$b'(x) = b''(x) = 0. \tag{6.16}$$

We easily obtain $\alpha = 1$ as the critical value of α. This defines the critical temperature.

Now consider $\delta \neq 0$, with x_0 metastable and x_2 stable. The expected time to reach x_2, given that $\tilde{x}(0) = x$ satisfies

$$-1 = \frac{a}{2} T_{xx} + bT_x, \tag{6.17}$$

$$T(x_2) = 0, \quad \lim_{x \to -\infty} T(x) < \infty \tag{6.18}$$

with $a(x)$ and $b(x)$ given by (6.13) and (6.11). Define the relaxation rate from the metastable to stable state by

$$k = \frac{1}{T(x_0)}. \tag{6.19}$$

We can calculate the relaxation rate k for all values of N. The method of Griffiths et al.[2]) broke down for large N. The result given here will be valid for all values of N.

It can be shown that the two results are equivalent for small N.

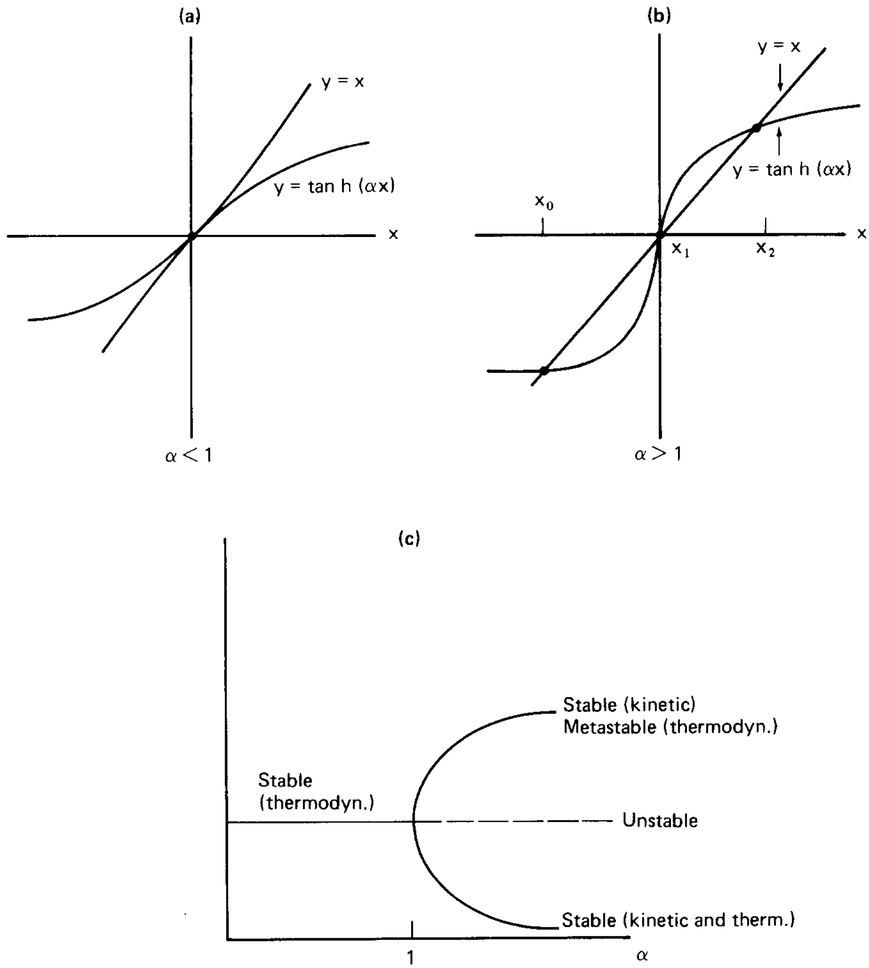

Fig. 9. Mean field ferromagnet.

7. Relaxation of a critical harmonic oscillator

The application in section 6 did not use the theory of section 4, but the one in this section does. Consider a Duffing oscillator

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v, \tag{7.1}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = (-\hat{k}(\varphi)x - \alpha x^3 - \gamma v) + \sqrt{2\epsilon}\,\frac{\mathrm{d}\tilde{y}}{\mathrm{d}t}. \tag{7.2}$$

Assume that $\hat{k}(\varphi) = 0$ for some critical value of φ and that $\hat{k}(\varphi) \geq 0$ for all

φ. The mean motion of the oscillator is given by

$$\dot{x} = v, \tag{7.3}$$

$$\dot{v} = -\hat{k}x - \alpha x^3 - \gamma v. \tag{7.4}$$

When $\alpha > 0$, the origin is the only real steady state. The matrix

$$B = (b^i_{,j})|_{0,0} = \begin{pmatrix} 0 & 1 \\ -\hat{k} & -\gamma \end{pmatrix} \tag{7.5}$$

has eigenvalues and eigenvectors

$$\lambda_\pm = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\hat{k}}}{2}, \quad e_\pm = \begin{pmatrix} 1 \\ \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4\hat{k}}}{2} \end{pmatrix} \tag{7.6}$$

Let $\bar{T}(x, v)$ be the expected time to enter a neighborhood of the origin, given $\tilde{x}(0) = x$, $\tilde{v}(0) = v$. Then

$$-1 = \epsilon \bar{T}_{vv} + v\bar{T}_x - (\hat{k}x + \alpha x^3 + \gamma v)\bar{T}_v. \tag{7.7}$$

At the critical value ϕ_c, eq. (7.7) becomes

$$-1 = \epsilon \bar{T}_{vv} + v\bar{T}_x - (\alpha x^3 + \gamma v)\bar{T}_v. \tag{7.8}$$

Since the origin is a critical type steady state, the theory of section 4 applies. The leading term in the asymptotic solution of (7.8) is

$$\bar{T}(x) \sim g^0 Q(\psi(x, v)/\epsilon^{1/4}, 0, 0, 1/\epsilon^{1/2}, 0, 0) + k^0(x) + \mathcal{O}(\epsilon^{3/4}). \tag{7.9}$$

Eqs. (4.27) and (4.29) become

$$v\psi_x - (\alpha x^3 + \gamma v)\psi_v + \psi_v^2 \psi^3 = 0, \tag{7.10}$$

$$vk_x^0 - (\alpha x^3 + v)k_v^0 - g^0 \frac{1}{2}\psi_v^2 = -1. \tag{7.11}$$

In order to keep ψ regular at $(0, 0)$, we set $\psi = 0$ there. In order to solve (7.10) by the method of characteristics, we need initial data for ψ_x and ψ_v. If (7.10) is differentiated with respect to v and evaluated at $(0, 0)$, we obtain

$$\psi_x - \gamma\psi_v = 0 \quad \text{at } (0, 0). \tag{7.12}$$

When (7.10) is differentiated three times with respect to x and evaluated at $(0, 0)$, we obtain

$$\psi_x^3 \psi_v^2 = \alpha/\gamma. \tag{7.13}$$

Thus we obtain, at $(0, 0)$

$$\psi_x = (\alpha\gamma)^{1/5}, \quad \psi_v = (\alpha^{1/5}) \cdot (\gamma^{4/5})^{-1}. \tag{7.14}$$

Higher derivatives are evaluated in a similar fashion. Thus, we can specify an ellipse around the origin:

$$N = \{(x, v): \psi(x, v) = \delta\}. \tag{7.15}$$

We set $Q(\delta/\epsilon^{1/4}, 0, 0, 1/\epsilon^{1/2}, 0, 0) = Q'(\delta/\epsilon^{1/4}, 0, 0, 1/\epsilon^{1/2}, 0, 0) = 0$ when integrating (4.26). We also set $k(x, v) = 0$ if $(x, v) \in N$.

At the origin, (7.11) becomes

$$g^0 = \frac{2}{\gamma\psi_v^2} = \frac{2}{\alpha^{2/5}}\gamma^{3/5}, \tag{7.16}$$

which determines the value of g^0. Then, on deterministic trajectories we have

$$\frac{dk^0}{dt} = -1 + \frac{g^0\gamma\psi_v^2}{2} \tag{7.17}$$

with the initial data given above. Eq. (7.10) can now be solved by the method of characteristics, so that the leading term in the asymptotic solution is known.

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